PHYS 705: Classical Mechanics

Kepler Problem: Geometry of Kepler Orbits

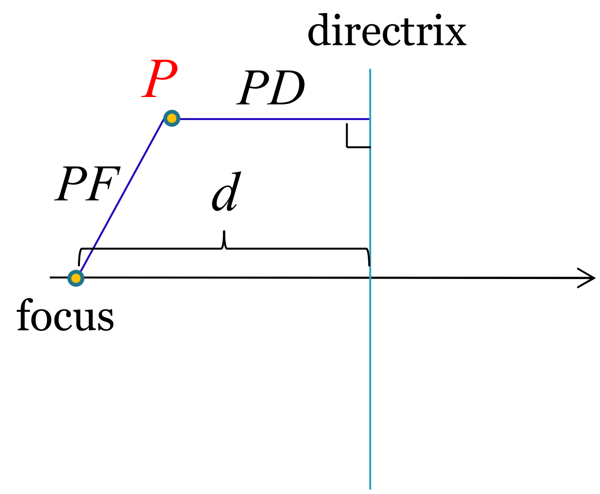
Focus-Directrix Formulation

In the following, we will study the geometry of the Kepler orbits

→ by considering the locus of points described by the Focus-Directrix formulation.

- Pick a focus
- Pick a directrix – a vertical line a distance d away
- Then, consider the set of points $\{P\}$ that satisfy,

$$PF = \varepsilon PD$$



PF = distance from P to focus
PD = distance from P to directrix

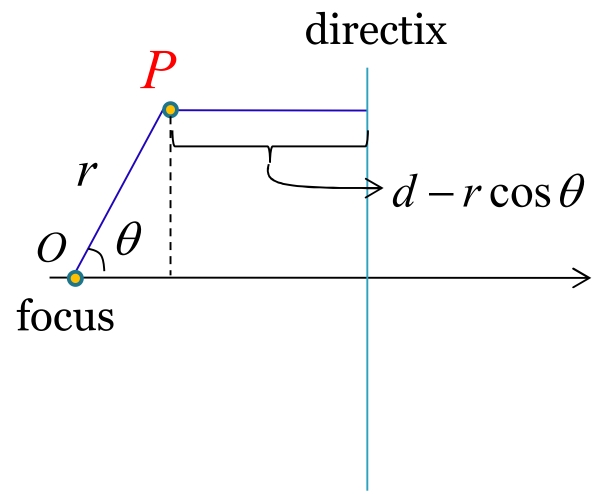
ε is the eccentricity

Focus-Directrix Formulation

Now, express this in polar coordinate with the focus as the origin:

$$PF = r$$

$$PD = d - r\cos\theta$$



Then, the condition $PF = \varepsilon PD$ gives,

$$r = \varepsilon\left(d - r\cos\theta\right)$$

Solving for r, we have,

$$r = \varepsilon d - \varepsilon r\cos\theta$$

$$r = \frac{\varepsilon d}{1 + \varepsilon\cos\theta}$$

Comparing with our previous Kepler orbit eq,

$$r = \frac{\alpha}{1 + \varepsilon\cos\theta}$$

→ They are the same with

$$\alpha = \varepsilon d \quad or \quad d = \alpha/\varepsilon$$

Focus-Directrix Formulation: Hyperbola

Recall the physical parameters for the orbit:

$$\alpha = \frac{l^2}{mk} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$$

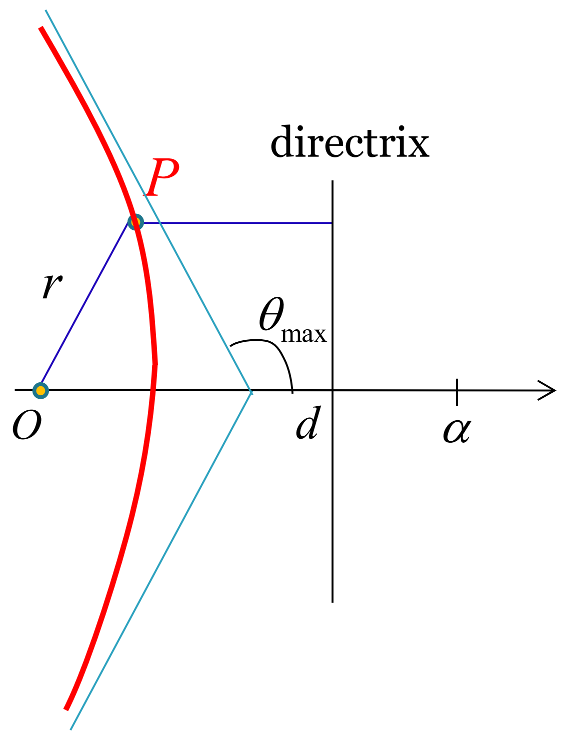
(Note: We have $\varepsilon \geq 0$ for physical orbits.)

Case 1: $E > 0 \quad \rightarrow \quad \varepsilon > 1$

The directrix is at $d = \dfrac{\alpha}{\varepsilon} < \alpha$

We have, $r = \dfrac{\varepsilon d}{1 + \varepsilon \cos\theta} = \dfrac{d}{1/\varepsilon + \cos\theta}$

→ Since $\varepsilon > 1, \quad 1/\varepsilon < 1$, the denominator can be zero for some angle $\theta_{\max} < \pi$ → r can increase without bound @ $\pm\theta_{\max}$ → This is a hyperbola !



Note: $\varepsilon \rightarrow \infty, d \rightarrow 0$

hyperbola → line at focus

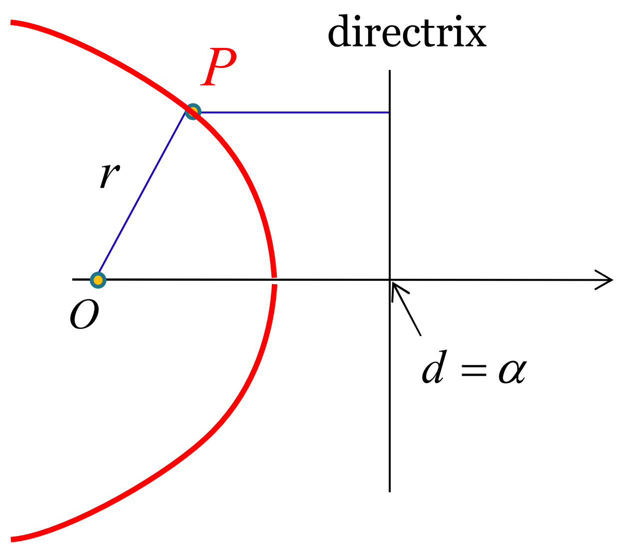
Focus-Directrix Formulation: Parabola

Case 2: $E = 0 \quad \rightarrow \quad \varepsilon = 1 \qquad \varepsilon = \sqrt{1 + \dfrac{2El^2}{mk^2}}$

The directix is at $d = \dfrac{\alpha}{\varepsilon} = \alpha$

Again, we have, $r = \dfrac{d}{1/\varepsilon + \cos\theta} = \dfrac{d}{1 + \cos\theta}$

→ Once again, the denominator can be zero at $\theta_{max} = \pm\,\pi$ → r can increase without bound at @ $\pm\pi$ → This is a parabola !



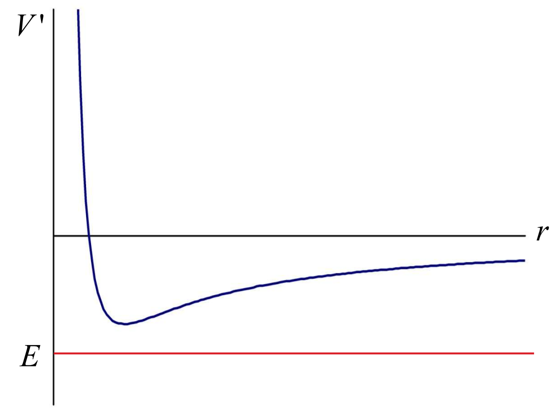
Focus-Directrix Formulation: Closed Orbit

Case 3: $E < 0$ There are three different sub-cases here.

recall $\varepsilon = \sqrt{1 + \dfrac{2El^2}{mk^2}}$

case 3a: $E < -\dfrac{mk^2}{2l^2}$ ⇨ ε is imaginary

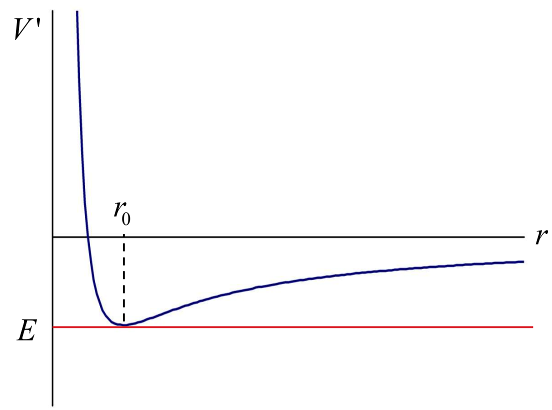
No orbits !



case 3b: $E = -\dfrac{mk^2}{2l^2}$ ⇨ $\varepsilon = 0$

$$r_0 = \frac{\alpha}{1+\varepsilon\cos\theta} = \alpha = \frac{l^2}{mk}$$

A circular orbit



Focus-Directrix Formulation: Circular Orbit

For circular orbits, we can combine the equations for E and r_o into one relation:

$$E = -\frac{mk^2}{2l^2} \quad \oplus \quad r_0 = \frac{l^2}{mk} \quad \Rightarrow \quad \boxed{E = -\frac{k}{2r_0}}$$

We can also get this from the Virial Theorem:

$$\overline{T} = -\frac{1}{2}\overline{V} = -\frac{1}{2}\left(-\frac{k}{r_0}\right) = \frac{k}{2r_0} \qquad \text{(at circular orbit } r = r_0\text{)}$$

$$E = T + V = \overline{T} + \overline{V} = \frac{k}{2r_0} - \frac{k}{r_0} = -\frac{k}{2r_0}$$

Focus-Directrix Formulation: Ellipse

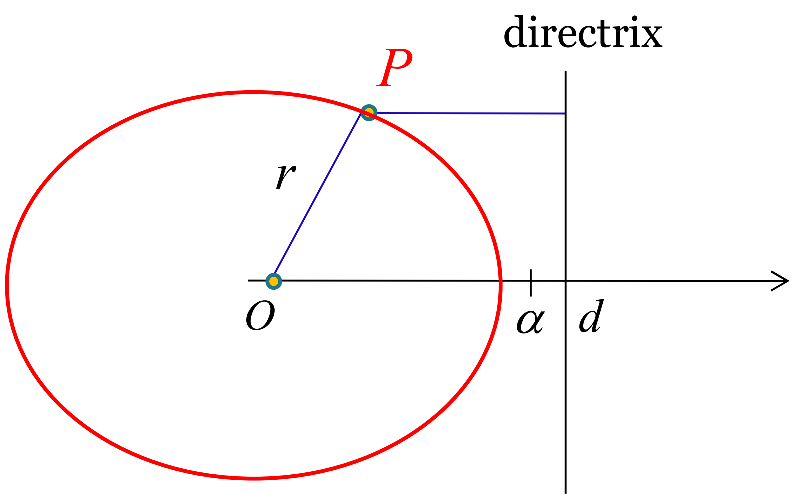
case 3c: $-\frac{mk^2}{2l^2} < E < 0 \Rightarrow 0 < \varepsilon < 1$

recall $\varepsilon = \sqrt{1 + \frac{2El^2}{mk^2}}$

The directrix is at $d = \frac{\alpha}{\varepsilon} > \alpha$ as indicated

Again, we have, $r = \frac{d}{1/\varepsilon + \cos\theta}$

→ Since $0 < \varepsilon < 1$, $1/\varepsilon > 1$, the denominator *cannot* be zero and *r* is bounded.



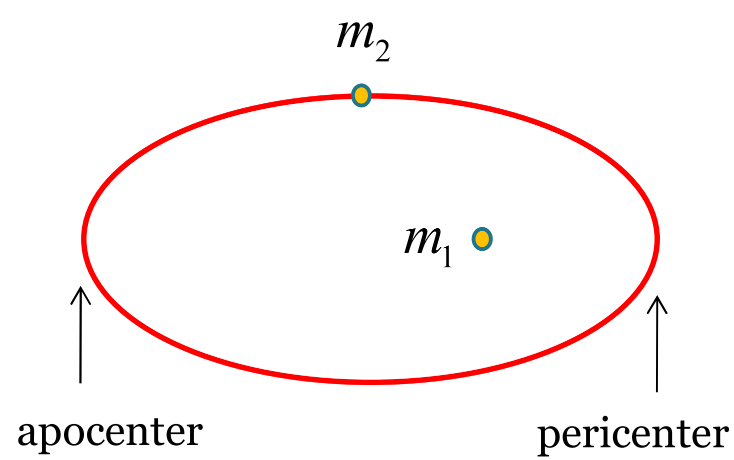
An ellipse

O is at one of the focii

Some Common Terminology

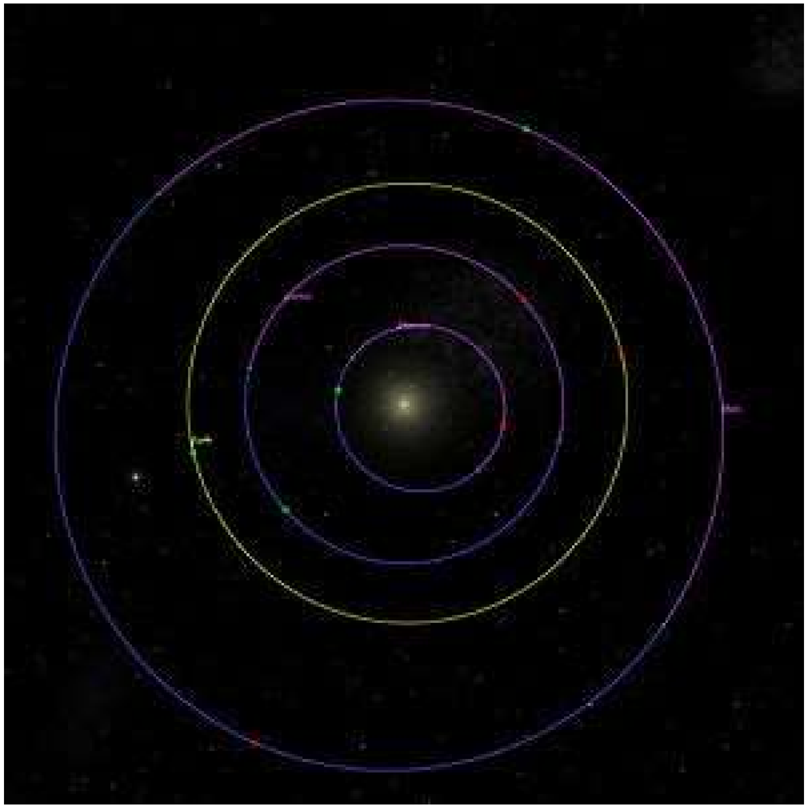
- Pericenter: the point of closest approach of mass 2 wrt mass 1
  - → Perigee refers to orbits around the earth
  - → Perihelion refers to orbits around the sun

- Apocenter: the point of farthest excursion of mass 2 wrt mass 1
  - → Apogee refers to orbits around the earth
  - → aphelion refers to orbits around the sun

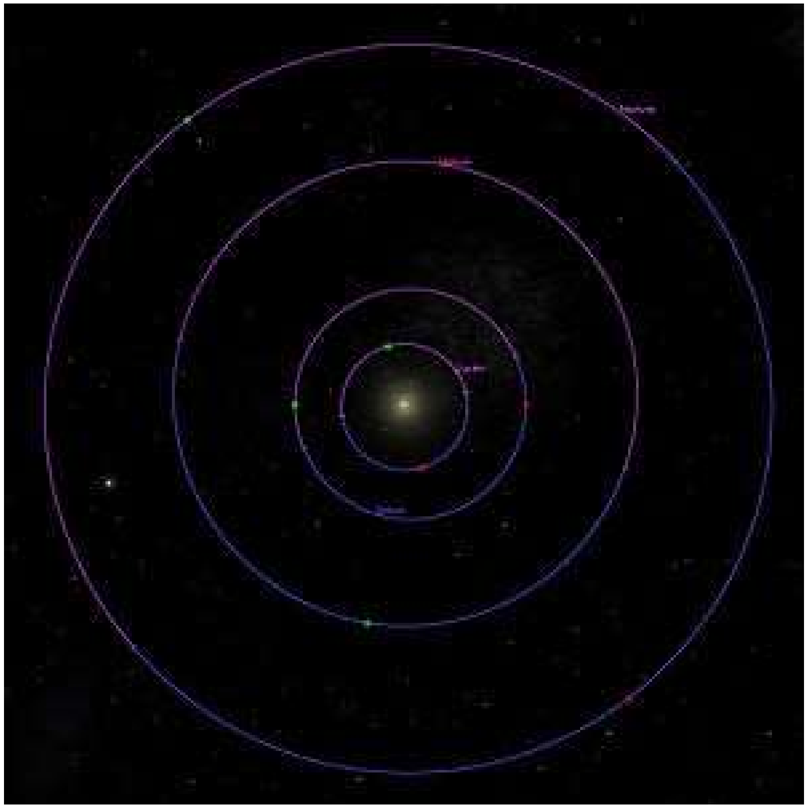


(with m_1 fixed in space at one of the foci)

Perihelion and aphelion of Planets in Solar System



Inner Planets



Outer Planets

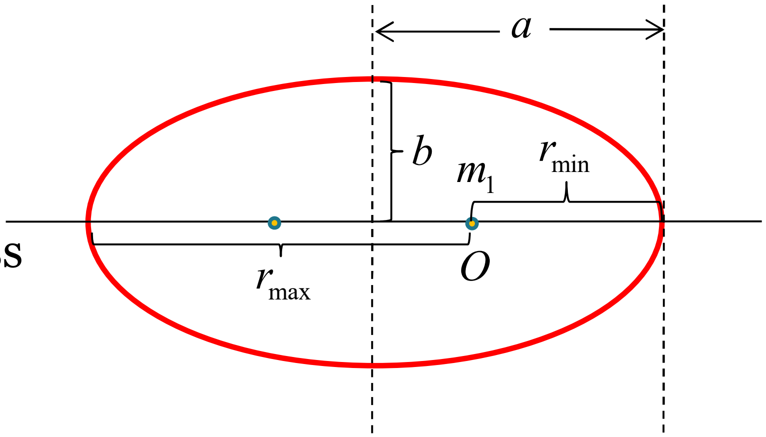
Focus-Directrix Formulation: Ellipse

Now, we are going to look at the elliptic case $(0 < \varepsilon < 1)$ closer:

(trying to relate geometry to physical parameters)

- The origin is with m_1 fixed in space at one of the foci of the ellipse

- The extremal distances of the reduced mass (or m_2) away from O (or m_1) are given by:



$$r_{min} = \left.\frac{d\varepsilon}{1+\varepsilon\cos\theta}\right|_{\theta=0} = \frac{d\varepsilon}{1+\varepsilon} = \frac{\alpha}{1+\varepsilon}$$ perihelion $(v_\theta \text{ largest})$

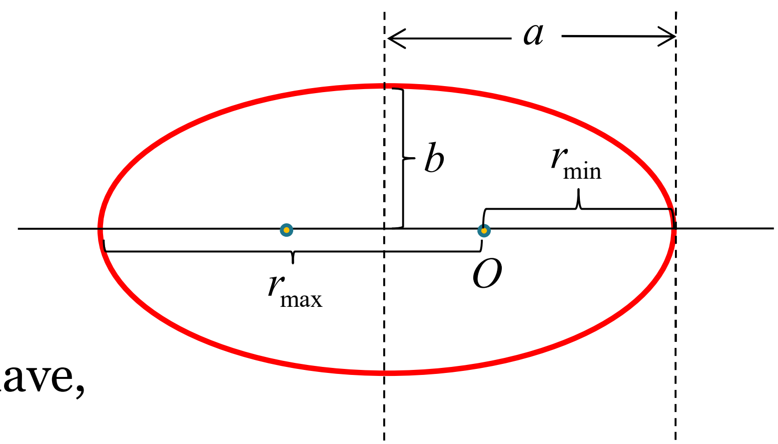
$$r_{max} = \left.\frac{d\varepsilon}{1+\varepsilon\cos\theta}\right|_{\theta=\pi} = \frac{d\varepsilon}{1-\varepsilon} = \frac{\alpha}{1-\varepsilon}$$ aphelion $(v_\theta \text{ smallest})$

Focus-Directrix Formulation: Ellipse

- The semi-major axis a:

$$a = \frac{r_{max} + r_{min}}{2} = \frac{1}{2}\left(\frac{\alpha}{1-\varepsilon} + \frac{\alpha}{1+\varepsilon}\right)$$

$$= \frac{\alpha\left(1 \cancel{+\varepsilon} + 1 \cancel{-\varepsilon}\right)}{2\left(1-\varepsilon^2\right)} = \boxed{\frac{\alpha}{1-\varepsilon^2} = a}$$



Now, putting in the physical constants, we have,

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{l^2}{mk}\frac{1}{1-\left(1+\frac{2El^2}{mk^2}\right)}$$

$$= \frac{l^2}{mk}\left(-\frac{mk^2}{2El^2}\right) = -\frac{k}{2E}$$

Thus, the semi-major axis depends on E and k only !

$$a = -\frac{k}{2E}$$

Focus-Directrix Formulation: Ellipse

Goldstein does this differently:

Start with the energy equation: $E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r}$

At the apsides, $\dot{r} = 0$, so $E - \frac{l^2}{2mr^2} + \frac{k}{r} = 0$ (@ apsides)

Write it as a quadratic equation in r:

$$r^2 + \frac{k}{E}r - \frac{l^2}{2mE} = 0$$

(this is an equation for the apsides)

Calling the two solutions for the apsides as: r_1 and r_2, we can write:

$$r^2 - (r_1 + r_2)r + (r_1 r_2) = 0$$

Recall $E < 0$ so $a > 0$

Comparing the two eqs, one immediately gets:

$$r_1 + r_2 = -\frac{k}{E} \xrightarrow[a=(r_1+r_2)/2]{} 2a = -\frac{k}{E} \Rightarrow a = -\frac{k}{2E}$$

Focus-Directrix Formulation: Ellipse

Side Notes:

→ The fact that a is a function of E only is an important assumption in the Bohr's model for the H-atom.

→ $a = -\dfrac{k}{2E}$ agrees with $r_0 = -\dfrac{k}{2E}$ in the limit for circular orbits.

→ Eccentricity can be expressed in terms of a:

Recall $\varepsilon = \sqrt{1 + \dfrac{2El^2}{mk^2}}$, substitute $E = -\dfrac{k}{2a}$

$$\Rightarrow \quad \varepsilon = \sqrt{1 + \frac{2l^2}{mk^2}\left(\frac{-k}{2a}\right)} = \sqrt{1 - \frac{l^2}{mak}}$$

(note: only true for ellipses and circles)

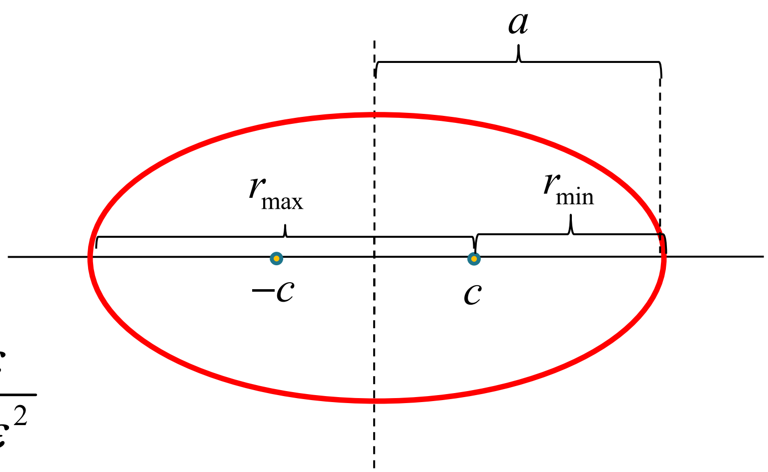
or $l^2/mk = \alpha = a\left(1 - \varepsilon^2\right)$

Focus-Directrix Formulation: Ellipse

Side Notes: The two focii are located at $\pm c = \pm \varepsilon a$

To show, start with:

$$c = \frac{r_{\max} - r_{\min}}{2}$$



Substituting $r_{\max}, r_{\min}$ in, we have,

$$c = \frac{r_{\max} - r_{\min}}{2} = \frac{1}{2}\left(\frac{\alpha}{1-\varepsilon} - \frac{\alpha}{1+\varepsilon}\right) = \frac{\alpha\varepsilon}{1-\varepsilon^2}$$

From previous page, we have $a = \dfrac{\alpha}{1-\varepsilon^2}$

This gives our result: $c = \dfrac{\varepsilon\alpha}{1-\varepsilon^2} = \varepsilon a$

Focus-Directrix Formulation: Ellipse

Side Notes:

→ The semi-minor axis b for an ellipse can be expressed as $b = a\sqrt{1-\varepsilon^2}$

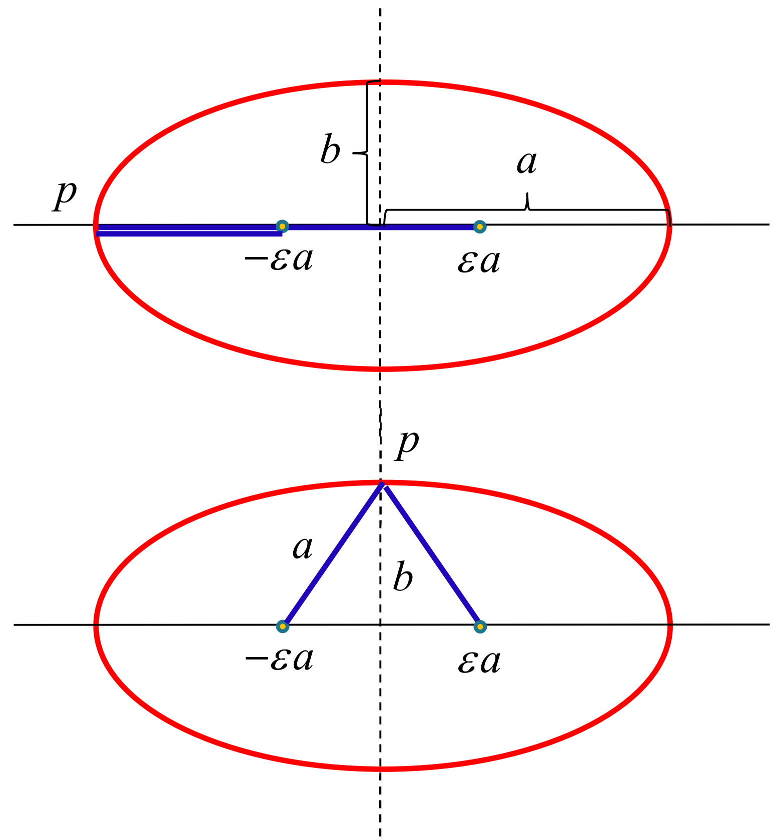
Recall that if one draws a line from one focus to any point p on the ellipse and back to the other focus, the length is fixed so that

$L = 2a$ (blue line)

But, we can also calculate it as,

$$L = 2a = 2\sqrt{b^2 + (\varepsilon a)^2}$$

⇒ $$\boxed{b = a\sqrt{1-\varepsilon^2}}$$



Focus-Directrix Formulation: Summary

Summary on conic sections for the Kepler orbits:

$$\varepsilon > 1 \qquad E > 0 \qquad \textit{hyperbola}$$

$$\varepsilon = 1 \qquad E = 0 \qquad \textit{parabola}$$

$$0 < \varepsilon < 1 \qquad E < 0 \qquad \textit{ellipse}$$

$$\varepsilon = 0 \qquad E = -\frac{mk^2}{2l^2} \qquad \textit{circle}$$

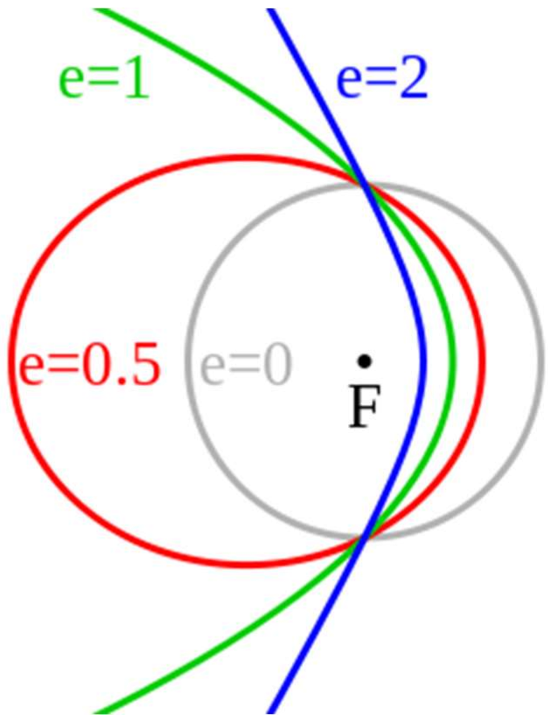
(ellipse and circle:) $$E = -\frac{k}{2a} \qquad l^2/mk = \alpha = a\left(1-\varepsilon^2\right)$$

$$r = \frac{\alpha}{1+\varepsilon\cos\theta}$$

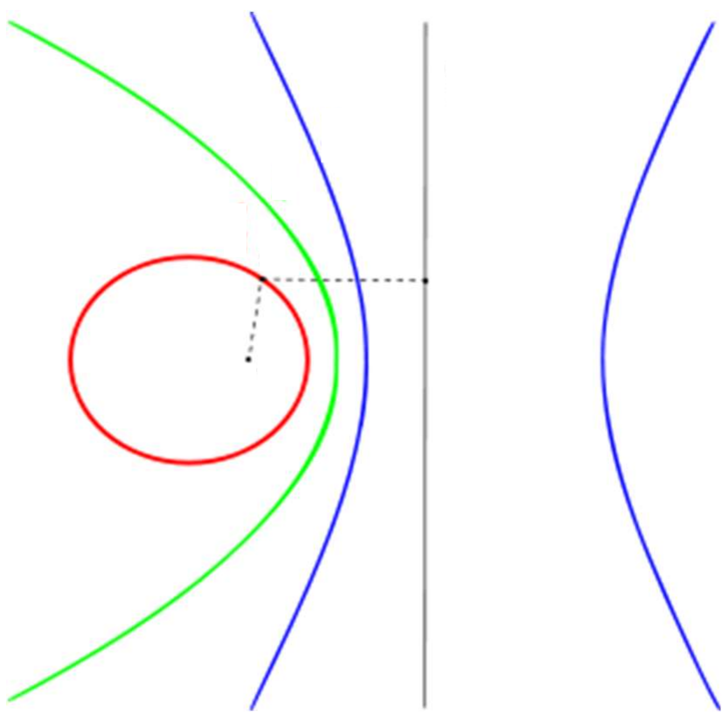
$$\alpha = \frac{l^2}{mk} \qquad \varepsilon = \sqrt{1+\frac{2El^2}{mk^2}}$$

Focus-Directrix Formulation: Summary

Kepler orbits with different energies $\left(E \leftrightarrow \varepsilon\right)$ but with the same angular momentum $\left(l \leftrightarrow \alpha\right)$



Focus-Directrix Formulation: Summary



A typical family of conic sections

In this picture, the directrix is fixed at one location so that d is the SAME for all orbits.

But, α and ε are different.

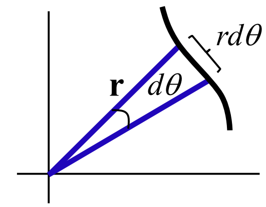
Recall that $d = \alpha / \varepsilon$ with

$$\alpha \leftrightarrow l \qquad \varepsilon \leftrightarrow E$$

So, d being the same → for diff E means l must also be different.

Kepler's 3rd Law: Period of an Elliptical Orbit

Recall that $dA = \frac{1}{2} r^2 d\theta$ and $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$



Combining this with the angular momentum equation: $l = mr^2 \dot{\theta}$

$$\frac{dA}{dt} = \frac{r^2}{2}\left(\frac{l}{mr^2}\right) = \frac{l}{2m} \quad \Longrightarrow \quad dt = \frac{2m}{l} dA$$

Integrating this *dt* over one round trip around the ellipse gives the period τ:

$$\tau = \int_0^\tau dt = \oint \frac{2m}{l} dA = \frac{2m}{l} A = \frac{2m}{l} \pi ab$$

Now, plug in our previous results for the semi-major and semi-minor axes:

$$\tau = \frac{2m}{l} \pi ab = \frac{2m}{l} \pi a \left(a\sqrt{1-\varepsilon^2} \right)$$

Kepler's 3rd Law: Period of an Elliptical Orbit

Square both sides, we have:

$$\tau^2 = \frac{4m^2}{l^2}\pi^2 a^4\left(1-\varepsilon^2\right)$$

Recall $\alpha = a\left(1-\varepsilon^2\right)$ and $\alpha = \frac{l^2}{mk}$ $\rightarrow$ $a\left(1-\varepsilon^2\right) = \frac{l^2}{mk}$

$\Rightarrow$ $\tau^2 = \frac{4m^2}{l^2}\pi^2 a^3\left(\frac{l^2}{mk}\right)$ $\Rightarrow$ $\boxed{\tau^2 = \frac{4\pi^2 m}{k}a^3}$ This is Kepler's 3rd Law

(Note: Recall that m is the reduced mass μ here and not $m_{1,2}$ so that Kepler original statement for orbits of planets around the sun is only an approximation which is valid for $\mu = m_\oplus m_{Sun}/(m_\oplus + m_{Sun}) \simeq m_\oplus$ and $\mu/k \simeq m_\oplus/Gm_\oplus m_{Sun} = 1/Gm_{Sun}$ so the proportionality $4\pi^2\mu/k \simeq 4\pi^2/Gm_{Sun}$ is approximately independent of $m_\oplus$.

Motion in Time: Eccentric Anomaly

Overview on how to analytically solve for the position of an elliptic orbit as a function of time: (can also be done for hyperbolic/parabolic orbits – hw)

(with the advent of computers, numerical methods replace this as the norm)

Recall from our EOM derivations, we get the following from E conservation:

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{l^2}{2mr^2}\right)}$$

Inverting r and t, we get:

$$t = \int_{r_0}^{r} dr \Bigg/ \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)} \qquad V(r) = -\frac{k}{r}$$

Motion in Time: True & Eccentric Anomaly

Our plan is to rewrite the equation in terms of $\psi \equiv$ *eccentric anomaly*

defined by: $r = a\left(1-\varepsilon\cos\psi\right)$ (basically, a change of variable $r \to \psi$)

As we will see, this can simplify the (analytical) integration for the EOM.

Before we continue onto the EOM, let get more familiar with this ψ

Recall that we have our Kepler orbit equation in polar coordinate in the reduced mass frame:

$$r = \frac{\alpha}{1+\varepsilon\cos\theta}$$

(choosing $\theta' = 0$ at perihelion)

Historically, θ is called the true anomaly and it can be shown that

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\left(\frac{\psi}{2}\right)$$

Motion in Time: True & Eccentric Anomaly

To derive this relation, we equate the two expressions for r:

$$a\left(1-\varepsilon\cos\psi\right)=\frac{\alpha}{1+\varepsilon\cos\theta}$$

Recall for an ellipse, the semi-major axis can be written as: $a=\dfrac{\alpha}{1-\varepsilon^2}$

$$\frac{\cancel{\alpha}}{1-\varepsilon^2}\left(1-\varepsilon\cos\psi\right)=\frac{\cancel{\alpha}}{1+\varepsilon\cos\theta}$$

$$1+\varepsilon\cos\theta=\frac{1-\varepsilon^2}{1-\varepsilon\cos\psi}$$

$$\cos\theta=\frac{1}{\varepsilon}\left(\frac{1-\varepsilon^2}{1-\varepsilon\cos\psi}-1\right)=\frac{1}{\cancel{\varepsilon}}\left(\frac{\cancel{1}-\varepsilon^{\cancel{2}}\cancel{-1}+\cancel{\varepsilon}\cos\psi}{1-\varepsilon\cos\psi}\right)$$

$$\cos\theta=\frac{\cos\psi-\varepsilon}{1-\varepsilon\cos\psi}$$

Motion in Time: True & Eccentric Anomaly

Now, evaluate the following two quantities,

$$1-\cos\theta = 1-\frac{\cos\psi-\varepsilon}{1-\varepsilon\cos\psi} = \frac{1-(1+\varepsilon)\cos\psi+\varepsilon}{1-\varepsilon\cos\psi} = \frac{(1+\varepsilon)(1-\cos\psi)}{1-\varepsilon\cos\psi}$$

$$1+\cos\theta = 1+\frac{\cos\psi-\varepsilon}{1-\varepsilon\cos\psi} = \frac{1+(1-\varepsilon)\cos\psi-\varepsilon}{1-\varepsilon\cos\psi} = \frac{(1-\varepsilon)(1+\cos\psi)}{1-\varepsilon\cos\psi}$$

Then , we divide these two expressions:

LHS: $$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos^2\theta/2+\sin^2\theta/2}{1+\cos^2\theta/2-\sin^2\theta/2} = \frac{2\sin^2\theta/2}{2\cos^2\theta/2} = \tan^2\theta/2$$

RHS: $$\frac{(1+\varepsilon)(1-\cos\psi)}{(1-\varepsilon)(1+\cos\psi)} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\tan^2\left(\frac{\psi}{2}\right)$$

Motion in Time: True & Eccentric Anomaly

Putting LHS = RHS and taking the square root, we then have,

$$\tan^2\left(\frac{\theta}{2}\right) = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\tan^2\left(\frac{\psi}{2}\right) \quad or \quad \boxed{\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\left(\frac{\psi}{2}\right)}$$

Going through one cycle around the orbit, we have these relations:

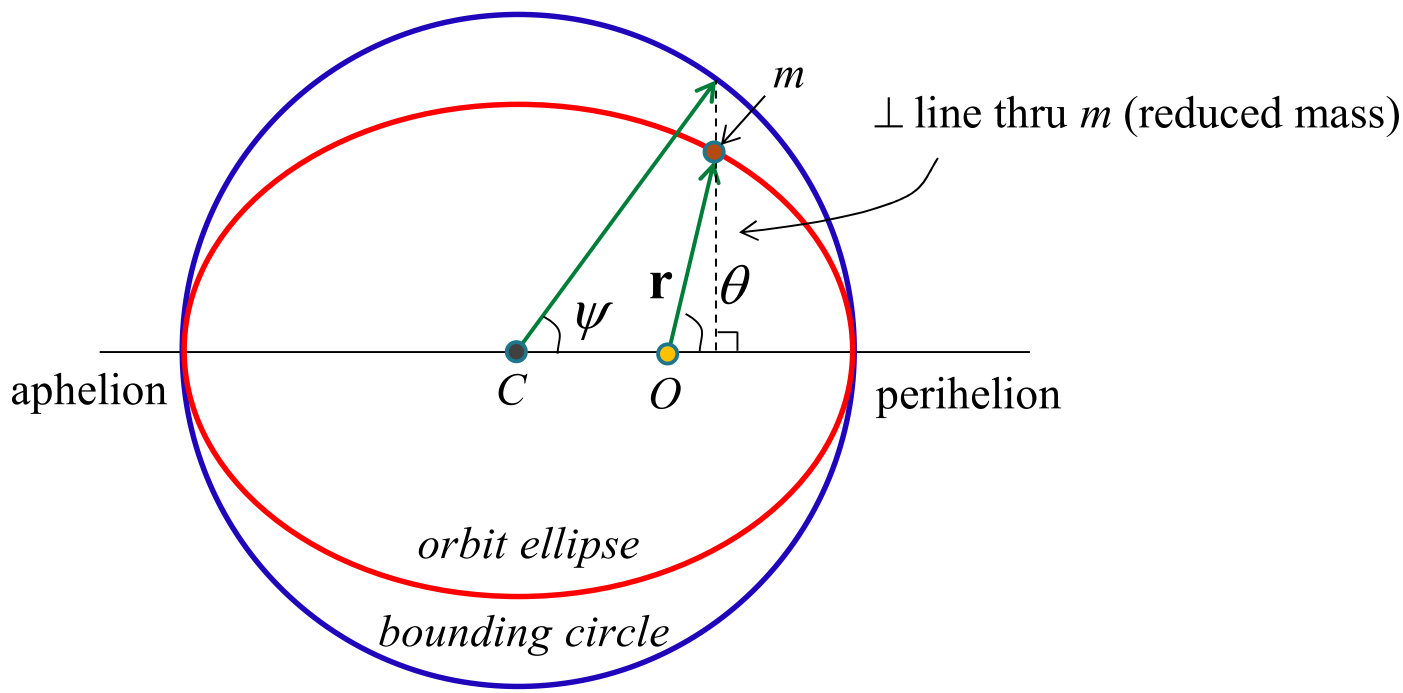
Starting at perihelion $\left[r_{\min} = a(1-\varepsilon)\right],\ \theta = 0 \quad \rightarrow \quad \psi = 0$

Reaching aphelion $\left[r_{\max} = a(1+\varepsilon)\right],\ \theta = \pi \quad \rightarrow \quad \psi = \pi$

Going back to perihelion $\left[r_{\min} = a(1-\varepsilon)\right],\ \theta = 2\pi \quad \rightarrow \quad \psi = 2\pi$

True & Eccentric Anomaly: Geometry

The true and eccentric anomaly are related geometrically as follows:



(O is at one of the focii of the orbit ellipse)

(C is the center of bounding circle)

Motion in Time: Eccentric Anomaly

Ok. Now, coming back to the time evolution of the orbit:

$$t = \int_{r_0}^{r} dr \Big/ \sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)}$$

Substituting the relations (on the right) into the denominators of the above eq, we have:

$$E = -\frac{k}{2a}, \quad \frac{l^2}{mk} = \alpha = a\left(1-\varepsilon^2\right)$$

$$\sqrt{\frac{2}{m}\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)} = \left[\frac{2}{m}\left(-\frac{k}{2a} + \frac{k}{r} - \left(\frac{k}{2r^2}\right)a\left(1-\varepsilon^2\right)\right)\right]^{1/2}$$

$$= \frac{1}{r}\sqrt{\frac{2k}{m}}\left[-\frac{r^2}{2a} + r - \frac{a\left(1-\varepsilon^2\right)}{2}\right]^{1/2}$$

Motion in Time: Eccentric Anomaly

Putting this back into the time integral, we have,

$$t = \sqrt{\frac{m}{2k}} \int_{r_0}^{r} rdr \Big/ \sqrt{r - \frac{r^2}{2a} - \frac{a\left(1-\varepsilon^2\right)}{2}}$$

(this is Eq. 3.69 in Goldstein)

Now, we substitute the eccentric anomaly into above:

→ By convention, we pick starting point r_o at perihelion $\psi = 0\ (\theta = 0)$

Then, limit of integration becomes: $\int_{r_o}^{r} dr \rightarrow \int_{o}^{\psi} d\psi$

→ Now, we need to do some more algebra to transform the integrant:

First, the change of variable $r = a\left(1 - \varepsilon \cos\psi\right)$ gives,

$dr = a\varepsilon \sin\psi\, d\psi$ and $rdr = a^2\varepsilon\left(1 - \varepsilon\cos\psi\right)\sin\psi\, d\psi$

Motion in Time: Eccentric Anomaly

→ Now, we need to transform the square-root term:

$$\sqrt{\ } = \left[a\left(1-\varepsilon\cos\psi\right) - \frac{a^2\left(1-\varepsilon\cos\psi\right)^2}{2a} - \frac{a\left(1-\varepsilon^2\right)}{2} \right]^{1/2}$$

$$= \left[\frac{2a^2\left(1-\varepsilon\cos\psi\right) - a^2\left(1-\varepsilon\cos\psi\right)^2 - a^2\left(1-\varepsilon^2\right)}{2a} \right]^{1/2}$$

$$= \left[\frac{a^2\left(1-\varepsilon\cos\psi\right)\left(2-\left(1-\varepsilon\cos\psi\right)\right) - a^2 + a^2\varepsilon^2}{2a} \right]^{1/2}$$

$$= \left[\frac{a^2\left(1-\varepsilon\cos\psi\right)\left(1+\varepsilon\cos\psi\right) - a^2 + a^2\varepsilon^2}{2a} \right]^{1/2}$$

Motion in Time: Eccentric Anomaly

$$\sqrt{\ } = \left[\frac{a^2\left(\cancel{1} - \varepsilon^2\cos^2\psi\right) \cancel{-a^2} + a^2\varepsilon^2}{2a} \right]^{1/2}$$

$$= \left[\frac{a}{2}\varepsilon^2\left(1-\cos^2\psi\right) \right]^{1/2}$$

$$= \sqrt{\frac{a}{2}}\varepsilon\sqrt{\sin^2\psi} = \sqrt{\frac{a}{2}}\varepsilon\sin\psi$$

Recall,

$$rdr = a^2\varepsilon\left(1-\varepsilon\cos\psi\right)\sin\psi\, d\psi$$

Now, putting all the pieces back together, we have,

$$t = \sqrt{\frac{m}{2k}}\int_{r_0}^{r}\frac{rdr}{\sqrt{\ }} = \sqrt{\frac{m}{2k}}\sqrt{\frac{2}{a}}\int_0^{\psi}\frac{a^2\cancel{\varepsilon}\left(1-\varepsilon\cos\psi'\right)\cancel{\sin\psi'}\,d\psi'}{\cancel{\varepsilon}\ \cancel{\sin\psi'}}$$

$$t = \sqrt{\frac{ma^3}{k}}\int_0^{\psi}\left(1-\varepsilon\cos\psi'\right)d\psi'$$

Motion in Time: Eccentric Anomaly

Carry through the integration, we have,

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} \left(1 - \varepsilon \cos \psi'\right) d\psi' = \sqrt{\frac{ma^3}{k}} \left(\psi - \varepsilon \sin \psi\right)$$

(Note the dramatic simplification using ψ)

Integrating this over ONE period of the elliptic orbit,

i.e., taking ψ from 0 to 2π, we have: $\tau = \sqrt{\frac{ma^3}{k}} 2\pi$

Again, we have Kepler's 3rd law:

$$\tau^2 = \frac{4\pi^2 m}{k} a^3$$

Motion in Time: Eccentric Anomaly

Defining the mean anomaly M as: $M = 2\pi \frac{t}{\tau}$ (M is observable)

→ We arrive at the **Kepler's Equation**:

$$M = \frac{2\pi}{\tau} t = 2\pi \sqrt{\frac{k}{4\pi^2 m a^3}} \sqrt{\frac{m a^3}{k}} (\psi - \varepsilon \sin \psi)$$

$$\boxed{M = \psi - \varepsilon \sin \psi}$$

Standard (non-numeric) procedure in solving celestial orbit equation:

1. Solve Kepler's equation → $\psi(t)$ [transcendental]
2. Use the $\psi \leftrightarrow \theta$ transformation to get back to the true anomaly $\theta(t)$.

Hohmann Transfer

The most economical method (minimum total energy expenditure) of changing among circular orbits in a Kepler system such as the Solar system.

Simplifying assumptions:

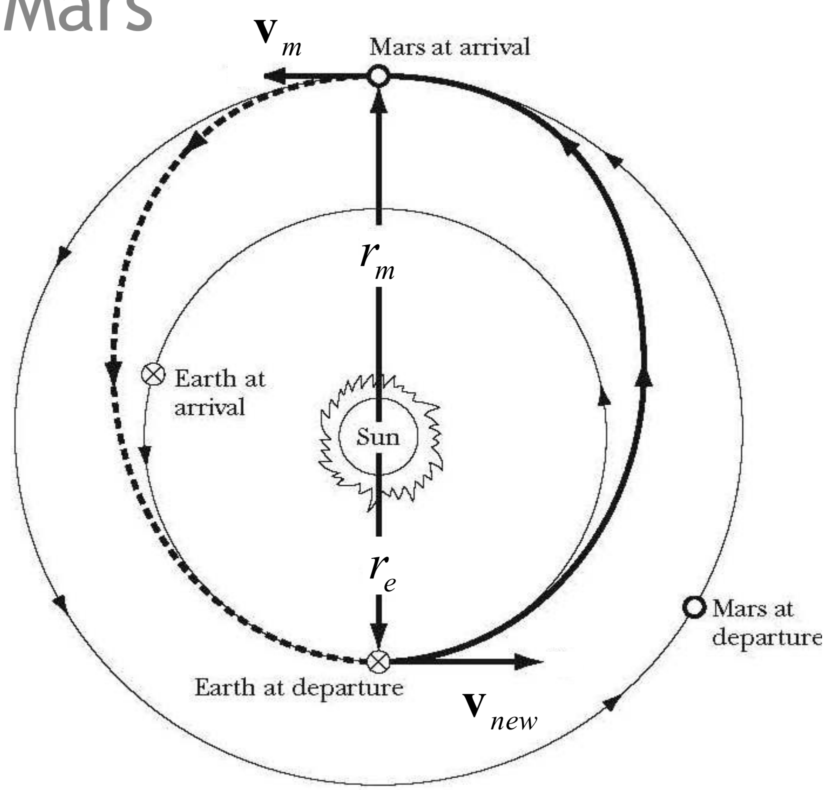
- Orbits are heliocentric (orbits with the Sun at the center of force)

  e.g. Orbit near Earth → Orbit near Mars
- Orbits are on the same orbital plane of the Sun
- $M_{sun} \gg m_{satellite}$
- Ignoring gravitational effects from other Planets
- Thrusts (two) from rockets change only eccentricity and energy of an orbit BUT not the direction of angular momentum

Hohmann Transfer: Earth to Mars

1. Start from Earth's "circular" orbit
2. Kick to go on transfer orbit (elliptic – **thick** line) with new speed v_{new}
3. Kick to slow down when arrived at Mars "circular" orbit with speed v_m

(Both E's and M's orbits are close to circular: $\varepsilon_{Earth} = 0.0167, \varepsilon_{Mars} = 0.0934$)



Note: 1. Both kicks are done at points of *tangency* (@ perihelion and aphelion of the ellipse) so that transfer can be accomplished with *speed-change* only.

2. Both (speed) kicks won't change the direction of **L** (stay on the same orbital plane).

3. Need to time operations so that Mars will actually be there at arrival.

Hohmann Transfer: Earth to Mars

Let calculate the required speed changes:

- For both circular and elliptic orbits,

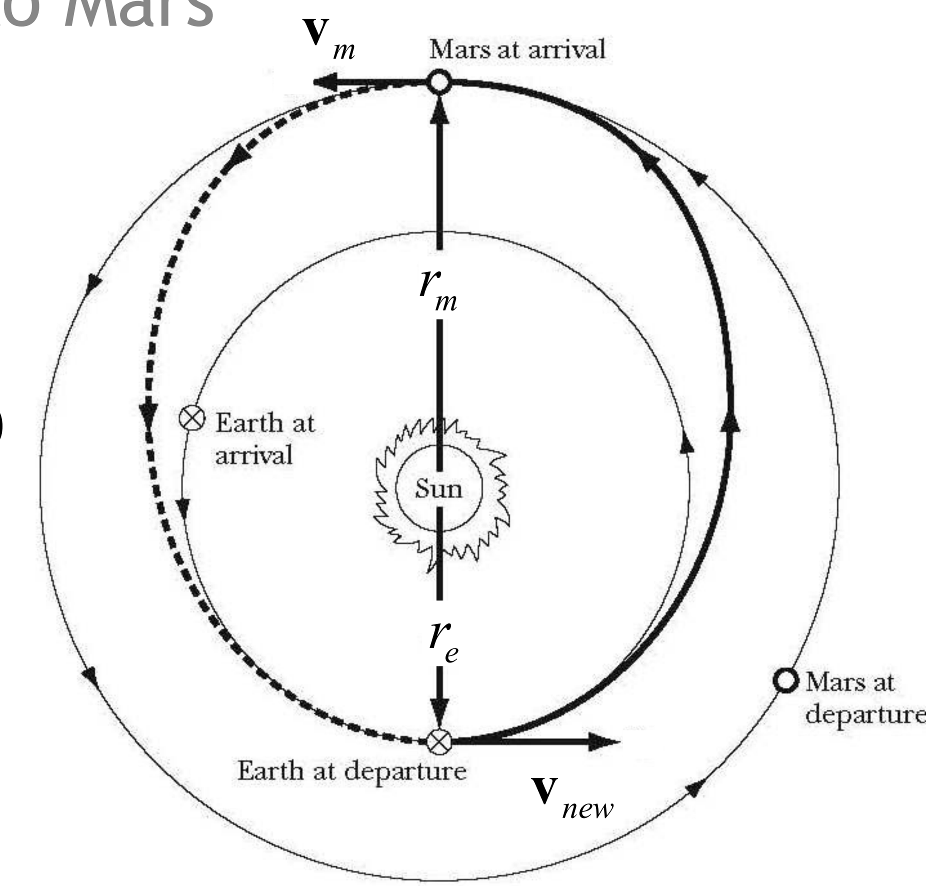
$$E = -\frac{k}{2a} \qquad (a \text{ semi-major axis})$$

- On Earth's orbit, we have

$$E = -\frac{k}{2r_e}$$

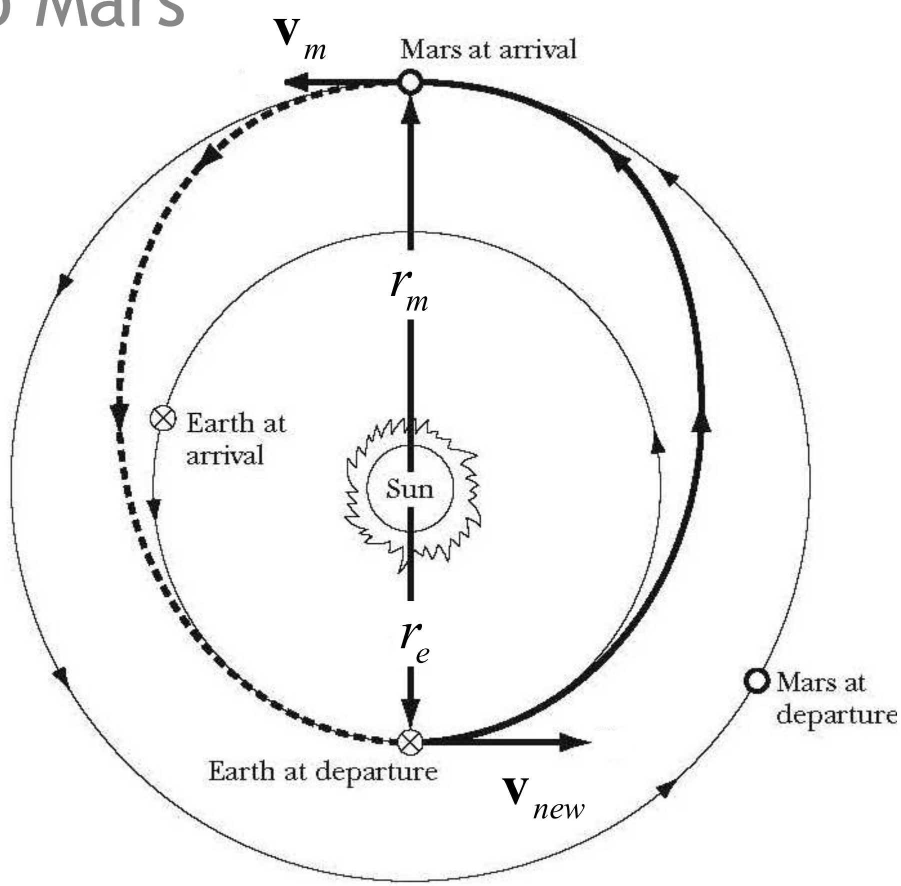
- Recall also, $E = \frac{1}{2}mv_e^2 - \frac{k}{r_e}$

- Combining with above gives, $v_e = \sqrt{\frac{k}{mr_e}}$



This is the Earth's orbital speed and the satellite will start with it as well.

Hohmann Transfer: Earth to Mars



Now, think of the position as the pericenter of the elliptic transfer orbit:

- The ellipse has its semi-major axis:

$$a = \frac{r_e + r_m}{2}$$

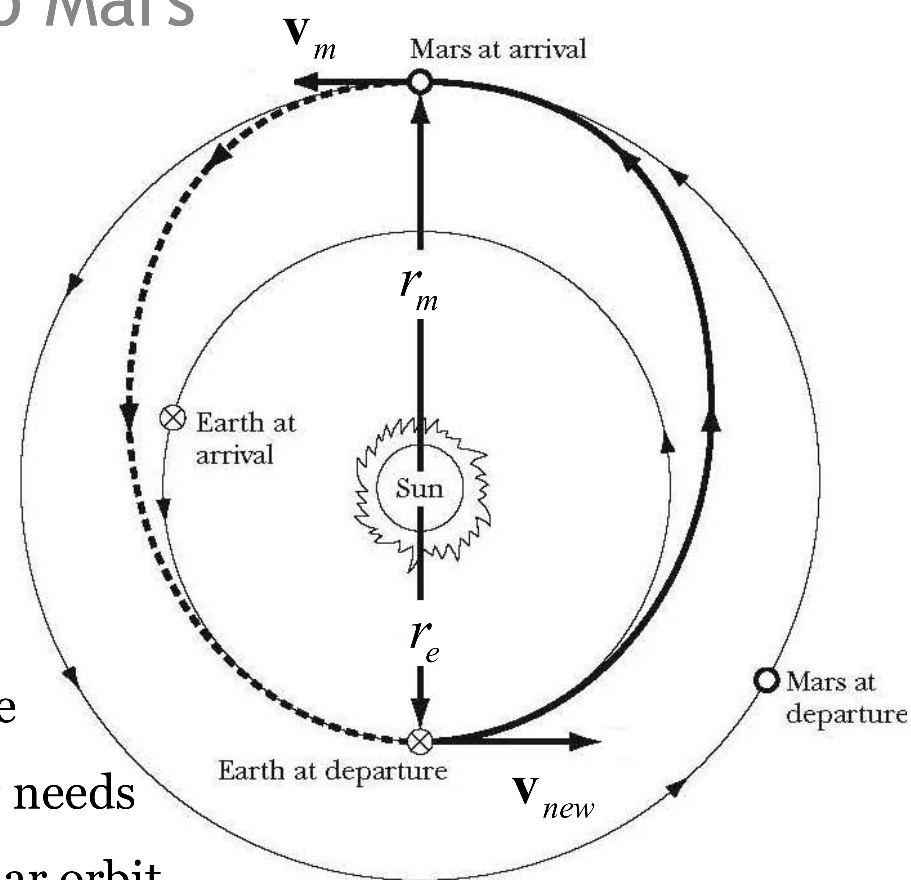
- So, for the satellite to be on this orbit, it needs to have energy:

$$E = -\frac{k}{r_e + r_m} = \frac{1}{2} m v_{new}^2 - \frac{k}{r_e}$$

- Solve for v_{new} ,

$$v_{new}^2 = \frac{2k}{m}\left(\frac{1}{r_e} - \frac{1}{r_e + r_m}\right) = \frac{2k}{m r_e}\left(\frac{r_m}{r_e + r_m}\right) \Rightarrow v_{new} = \sqrt{\frac{2k}{m r_e}\left(\frac{r_m}{r_e + r_m}\right)}$$

Hohmann Transfer: Earth to Mars



- So, to get the satellite onto the elliptic transfer orbit, we need to give a “kick” to speed it up (no change in direction)

$$\Delta v_1 = v_{new} - v_{earth}$$

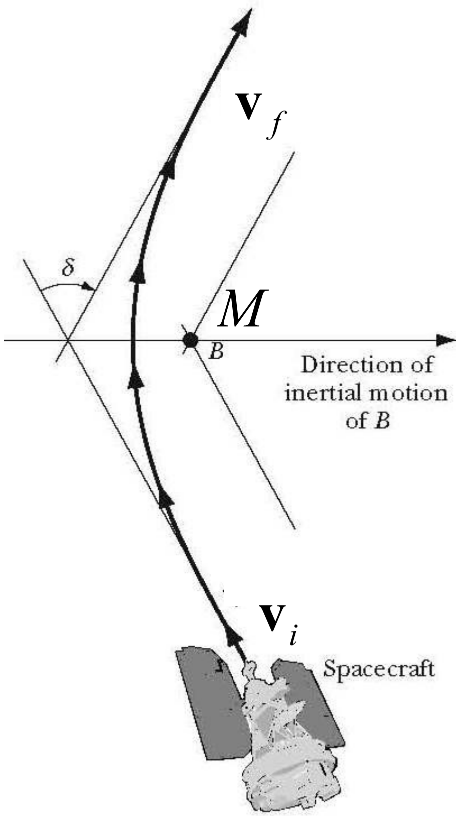
$$\Delta v_1 = \sqrt{\frac{2k}{mr_e}\left(\frac{r_m}{r_e + r_m}\right)} - \sqrt{\frac{k}{mr_e}}$$

- When satellite arrives at the aphelion of the transfer orbit (Mars' circular orbit), thruster needs to fire to slow down to get onto Mars's circular orbit,

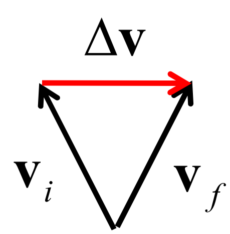
$$E = -\frac{k}{2r_m} = \frac{1}{2}mv_m^2 - \frac{k}{r_m} \quad \text{(on Mars's orbit)} \Rightarrow \quad v_{new} \to v_m = \sqrt{\frac{k}{mr_m}}$$

Slingshot Effect (qualitative discussion)

Let consider a satellite swinging by a Planet (hyperbolic orbit) in a frame co-moving with Planet *M*.

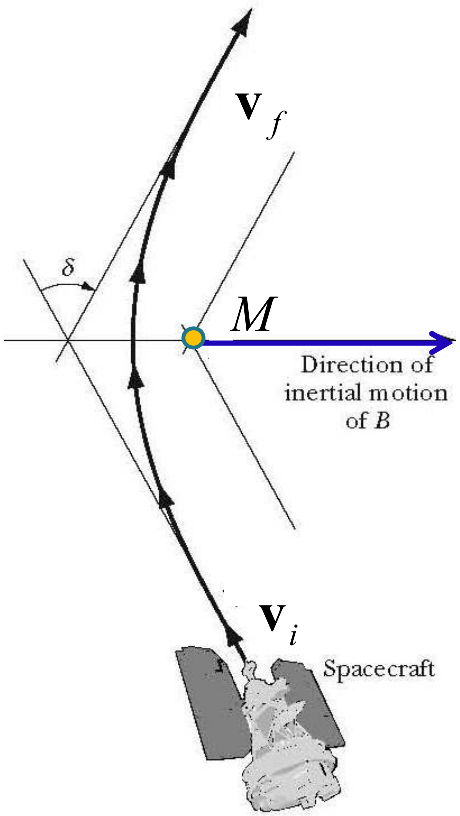


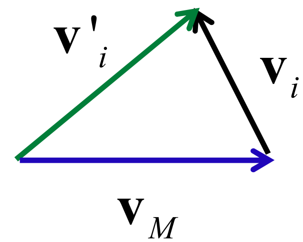
Note: the hyperbolic orbit is symmetric with respect to the apside → magnitude of **v** doesn't change, (before and after) only its direction.



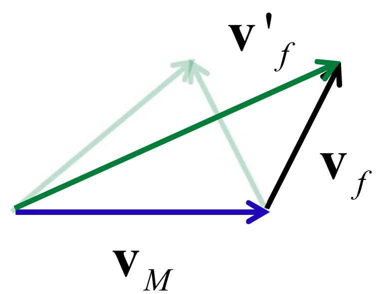
Slingshot Effect (qualitative discussion)

Now, if we consider the same situation in a “stationary” inertial reference frame (fixed wrt the Sun) in which the Planet M is moving to the right.





$\mathbf{v}'_i$ is the velocity of the satellite in the “stationary” frame.



$\mathbf{v}'_f$ is the velocity of the satellite in the “stationary” frame.

⇨ Note that $\mathbf{v}'_f$ has a larger magnitude than $\mathbf{v}'_i$ (with a boost in the direction of Planet M).

ISEE-3 Fly-by (1982-1985)

- ISEE-3 (International Sun-Earth Explorer 3) was originally parked in an orbit at the L1 Lagrange point to observe the Sun and the Earth.
- 1982: NASA decided to reprogram it so that it will go to explore the Giacobini-Zinner comet scheduled to visit the inner Planets in September of 1985.
- Design an orbit to get it from its parked orbit to the coming comet: cheaper than to design, build, and lunch a new satellite.

Some details: First burn $\Delta v < 10\, mph$

A total of 37 burns

Two close trips back to Earth and five flybys of the moon.

One pass within 75 miles of the Luna surface

Resulted in 20 min trip through the comet tail on 9/11/85.

ISEE-3 Fly-by (1982-1985)